

Least-Squares Methods for Geosciences: A Fair and Balanced Perspective

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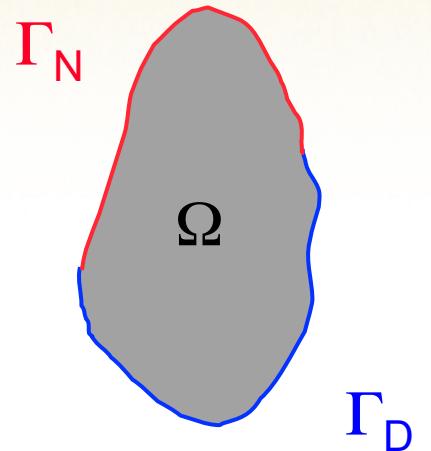
A Model Problem

$$\begin{aligned} -\nabla \cdot \mathbf{A} \nabla \phi + \gamma \phi &= f && \text{in } \Omega \\ \phi &= g && \text{on } \Gamma_D \\ \mathbf{n} \cdot \mathbf{A} \nabla \phi &= h && \text{on } \Gamma_N \end{aligned}$$

$$\begin{cases} \nabla \cdot \mathbf{u} + \gamma \phi = f \\ \mathbf{A}^{-1} \mathbf{u} + \nabla \phi = 0 \end{cases} \quad \text{in } \Omega$$

$$\begin{aligned} \gamma \in L^\infty(\Omega) &\rightarrow \begin{cases} \gamma \equiv 0 \\ \gamma \geq \gamma_0 > 0 \end{cases} \\ f \in L^2(\Omega) & \\ \mathbf{A} \in \mathbf{R}^{d \times d} &\rightarrow \frac{1}{\alpha} |\xi|^2 \leq \xi^T \mathbf{A} \xi \leq \alpha |\xi|^2 \end{aligned}$$

Equivalent first-order system



Domains of grad, curl and div

$$H_D(\Omega, \text{grad}) = H_D^1(\Omega) = \left\{ \varphi \in L^2(\Omega) \mid \nabla \varphi \in L^2(\Omega) \quad \& \quad \varphi = 0 \text{ on } \Gamma_D \right\}$$

$$L^2(\Omega) = \left\{ \varphi \mid \int_{\Omega} \varphi^2 d\Omega < \infty \right\}$$

$$H_D(\Omega, \text{curl}) = \left\{ \mathbf{v} \in \mathbf{L}^2(\Omega) \mid \nabla \times \mathbf{v} \in \mathbf{L}^2(\Omega) \quad \& \quad \mathbf{n} \times \mathbf{v} = 0 \text{ on } \Gamma_D \right\}$$

$$L^2(\Omega) = \left\{ \mathbf{v} \mid \int_{\Omega} |\mathbf{v}|^2 d\Omega < \infty \right\}$$

$$H_N(\Omega, \text{div}) = \left\{ \mathbf{v} \in \mathbf{L}^2(\Omega) \mid \nabla \cdot \mathbf{v} \in L^2(\Omega) \quad \& \quad \mathbf{n} \cdot \mathbf{v} = 0 \text{ on } \Gamma_N \right\}$$

Range of grad

$$H_D^0(\Omega, \text{curl}) = \left\{ \mathbf{v} \in \mathbf{L}^2(\Omega) \mid \mathbf{v} = \nabla \varphi \quad \text{for } \varphi \in H_D(\Omega, \text{grad}) \right\} \subset H_D(\Omega, \text{curl})$$



A Least-Squares Principle (LSP)

Artificial “energy” principle

$$\left. \begin{array}{l} \nabla \cdot \mathbf{u} + \gamma\phi = f \\ \mathbf{A}^{-1}\mathbf{u} + \nabla\phi = 0 \end{array} \right\} \Leftrightarrow J(\mathbf{u}, \phi; f) = \frac{1}{2} \left(\|\nabla \cdot \mathbf{u} + \gamma\phi - f\|_0^2 + \|\mathbf{A}^{-1/2}(\mathbf{u} + \mathbf{A}\nabla\phi)\|_0^2 \right) = 0$$

Optimization problem

$$\min_{\mathbf{v} \in H_N(\Omega, \text{div}); \psi \in H_D^1(\Omega)} J(\mathbf{v}, \psi; f)$$

Optimality system

$$\begin{aligned} (\nabla \cdot \mathbf{u} + \gamma\phi, \nabla \cdot \mathbf{v}) + (\mathbf{A}^{-1/2}\mathbf{u} + \mathbf{A}^{1/2}\nabla\phi, \mathbf{A}^{-1/2}\mathbf{v}) &= (f, \nabla \cdot \mathbf{v}) \quad \forall \mathbf{v} \in H_N(\Omega, \text{div}) \\ (\nabla \cdot \mathbf{u} + \gamma\phi, \gamma\psi) + (\mathbf{A}^{-1/2}\mathbf{u} + \mathbf{A}^{1/2}\nabla\phi, \mathbf{A}^{1/2}\nabla\psi) &= (f, \gamma\psi) \quad \forall \psi \in H_D^1(\Omega) \end{aligned}$$

To use least squares:

- ☺ Using C⁰ elements
- ☺ No inf-sup condition
- ☺ Solving SPD systems

We will show that:

- ⇒ Using **C⁰ elements** is not necessarily the best choice in LSFEM, and so it is arguably the **least-important advantage** attributed to least-squares methods
- ⇒ By using **other** elements least-squares acquire **additional conservation** properties
- ⇒ Surprisingly, this kind of least-squares turns out to be **related to mixed methods**

Not to use least squares:

- ☹ Conservation
- ☹ Conservation
- ☹ Conservation

Least-Squares Theory

Artificial “energy” norm

$$J(\mathbf{u}, \phi; 0) = \frac{1}{2} \left(\|\nabla \cdot \mathbf{u} + \gamma \phi\|_0^2 + \|\mathbf{A}^{-1/2}(\mathbf{u} + \mathbf{A} \nabla \phi)\|_0^2 \right) = |||(\mathbf{u}, \phi)|||^2$$

Norm equivalence

$$C_1 \left(\|\mathbf{u}\|_{div}^2 + \|\phi\|_1^2 \right) \leq |||(\mathbf{u}, \phi)|||^2 \leq C_2 \left(\|\mathbf{u}\|_{div}^2 + \|\phi\|_1^2 \right)$$

Bilinear form

$$Q_{LS}(\mathbf{u}, \phi; \mathbf{v}, \psi) = (\nabla \cdot \mathbf{u} + \gamma \phi, \nabla \cdot \mathbf{v} + \gamma \psi) + (\mathbf{A}^{-1/2} \mathbf{u} + \mathbf{A}^{1/2} \nabla \phi, \mathbf{A}^{-1/2} \mathbf{v} + \mathbf{A}^{1/2} \nabla \psi)$$

Inner-product equivalence

$$Q_{LS}(\mathbf{u}, \phi; \mathbf{v}, \psi) = \langle (\mathbf{u}, \phi); (\mathbf{v}, \psi) \rangle \quad \text{and} \quad Q_{LS}(\mathbf{u}, \phi; \mathbf{u}, \phi) = |||(\mathbf{u}, \phi)|||^2$$

Stability

$$C_1 \left(\|\mathbf{u}\|_{div}^2 + \|\phi\|_1^2 \right) \leq Q_{LS}(\mathbf{u}, \phi; \mathbf{u}, \phi)$$



coercivity

$$\text{continuity } \rightarrow \quad Q_{LS}(\mathbf{u}, \phi; \mathbf{v}, \psi) \leq C_2 \left(\|\mathbf{u}\|_{div}^2 + \|\phi\|_1^2 \right)^{1/2} \left(\|\mathbf{v}\|_{div}^2 + \|\psi\|_1^2 \right)^{1/2}$$



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C⁰ Conformity \Rightarrow Stability

Because coercivity is inherited on any proper subspace, the inclusions

$$\mathbf{V}_h \subset H(\Omega, \text{div}) \quad \& \quad S_h \subset H^1(\Omega) \quad (\text{hold for C}^0 \text{ elements!})$$

are sufficient for stability of LSFEM and quasi-optimal energy norm error estimates

Least-squares FEM

$$Q_{LS}(\mathbf{u}_h, \phi_h; \mathbf{v}_h, \psi_h) = (f, \nabla \cdot \mathbf{v}_h + \gamma \psi_h) \quad \forall (\mathbf{v}_h, \psi_h) \in \mathbf{V}_h \times S_h$$

Error estimates

$$\begin{aligned} \|\phi - \phi_h\|_1 + \|\mathbf{u} - \mathbf{u}_h\|_{\text{div}} &\leq C \inf_{(\mathbf{v}_h, \psi_h) \in \mathbf{V}_h \times S_h} \|\mathbf{u} - \mathbf{v}_h\|_{\text{div}} + \|\phi - \phi_h\|_1 \\ \|\phi - \phi_h\|_0 &\leq Ch \|\phi - \phi_h\|_1 \end{aligned}$$

Example Using C⁰ (nodal) P2 elements for both the **flux** and the **pressure** yields

$$\begin{aligned} \|\phi - \phi_h\|_1 + \|\mathbf{u} - \mathbf{u}_h\|_{\text{div}} &\leq Ch^2 (\|\mathbf{u}\|_3 + \|\phi\|_3) \\ \|\phi - \phi_h\|_0 &\leq Ch^3 \|\phi\|_3 \end{aligned}$$

This has led to the misconception that in least-squares methods **all variables “can” and should be approximated by the same, equal order C⁰ spaces**

C^0 Conformity $\not\Rightarrow$ Optimal L^2 Accuracy

Unfortunately...

$\mathbf{V}_h \subset H(\Omega, \text{div})$ & $S_h \subset H^1(\Omega)$ **is insufficient** for optimal L^2 convergence of \mathbf{v}_h !

LS vs BA	scalar		vector	
	L^2	H^1	L^2	$H(\text{div})$
P1	2.00	1.00	1.38	0.99
BA	2.00	1.00	2.00	1.00
P2	3.00	2.00	2.02	2.00
BA	3.00	2.00	3.00	2.00

Optimal convergence of \mathbf{v}_h in L^2 has been achieved in 2 ways



1) By using an augmented LS principle

Idea

$$\mathbf{u} + \nabla\phi = 0 \Rightarrow \nabla \times \mathbf{u} = 0$$

Augmented PDE

$$\begin{cases} \nabla \cdot \mathbf{u} + \gamma\phi = f \\ \mathbf{u} + \mathbf{A}\nabla\phi = 0 \end{cases} \& \nabla \times \mathbf{u} = 0 \text{ in } \Omega; \quad \phi = 0 \text{ on } \Gamma_D; \mathbf{u} \cdot \mathbf{n} = 0 \text{ on } \Gamma_N$$

Functional

$$J(\mathbf{u}, \phi; f) = \frac{1}{2} \left(\|\nabla \cdot \mathbf{u} + \gamma\phi - f\|_0^2 + \|\mathbf{A}^{-1/2}(\mathbf{u} + \mathbf{A}\nabla\phi)\|_0^2 + \|\nabla \times \mathbf{u}\|_0^2 \right)$$

Norm equivalence
(under some restrictions)

$$C_1 (\|\mathbf{u}\|_1^2 + \|\phi\|_1^2) \leq \|(\mathbf{u}, \phi)\| \leq C_2 (\|\mathbf{u}\|_1^2 + \|\phi\|_1^2)$$

Error estimate (P2/Q2)



$$\begin{aligned} \|\mathbf{u} - \mathbf{u}_h\|_1 + \|\phi - \phi_h\|_1 &\leq Ch^2 (\|\mathbf{u}\|_3 + \|\phi\|_3) \\ \|\mathbf{u} - \mathbf{u}_h\|_0 + \|\phi - \phi_h\|_0 &\leq Ch^3 (\|\mathbf{u}\|_3 + \|\phi\|_3) \end{aligned}$$

Drawbacks

- ⇒ For some regions H^1 may have infinite co-dimension in $H(\text{curl}) \cap H(\text{div})$ (Costabel).
- ⇒ A FEM subspace of $H(\text{curl}) \cap H(\text{div})$ is necessarily a subspace of H^1 .
- ⇒ FEM will not converge to solutions that are not in H^1 .



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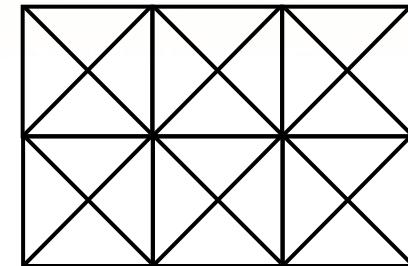
Or, 2) by using a special grid

The Grid Decomposition Property (GDP)

Fix, Gunzburger, Nicolaides, 1976

$$\forall \mathbf{v}_h \in V_h$$

$$\begin{cases} \mathbf{v}_h = \mathbf{w}_h + \mathbf{z}_h \\ \nabla \cdot \mathbf{z}_h = 0 \\ (\mathbf{w}_h, \mathbf{z}_h) = 0 \\ \|\mathbf{w}_h\|_0 \leq C(\|\nabla \cdot \mathbf{v}_h\|_{-1} + h\|\nabla \cdot \mathbf{v}_h\|_0) \end{cases}$$



The (only known) C^0 example

Theorem

GDP is *necessary and sufficient* for *stable* and *optimally accurate* mixed discretization of the **Least-Squares Principle (and the Mixed Method)**

Fix, Gunzburger, Nicolaides, Comp. Math with Appl. 5, pp.87-98, 1979

Using the criss-cross grid
and $S_h = \nabla \cdot \mathbf{V}_h$:



$$\|\mathbf{u} - \mathbf{u}_h\|_{div} + \|\phi - \phi_h\|_1 \leq Ch^1(\|\mathbf{u}\|_2 + \|\phi\|_2)$$

$$\|\mathbf{u} - \mathbf{u}_h\|_0 + \|\phi - \phi_h\|_0 \leq Ch^2(\|\mathbf{u}\|_2 + \|\phi\|_2)$$



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The Mixed Galerkin Connection

Lemma

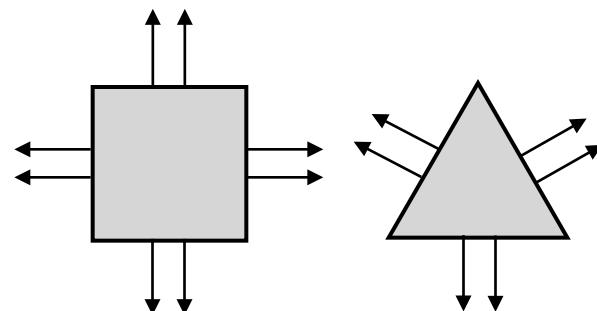
(Bochev, Gunzburger, SINUM 2005)

(V_h, S_h) satisfies the inf-sup condition $\Rightarrow V_h$ verifies GDP

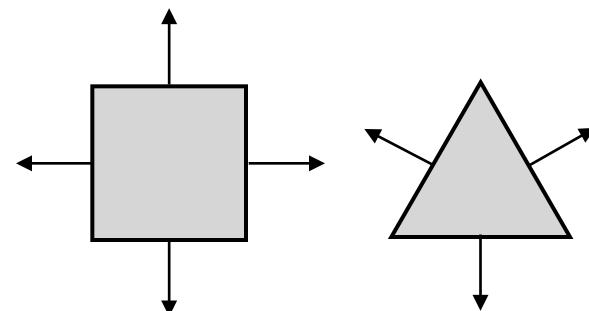
There are plenty of spaces that verify GDP

Except that they are **not C^0** (nodal)!

BDM(k) spaces $k \geq 1$



RT(k) spaces $k \geq 0$



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Mimetic Least-Squares

Using nodal C^0 elements for all variables is not the best choice!

Instead, pose the discrete LSP $\min_{\mathbf{v}_h \in D^h, \psi_h \in G^h} J(\mathbf{v}_h, \psi_h; f)$ on this pair of spaces:

$$D^h \subset H_N(\Omega, \text{div}) \quad \rightarrow \quad \text{any with GDP}$$

$$G^h \subset H_D^1(\Omega) \quad \rightarrow \quad \text{any that is } C^0$$

Theorem. For proof see Bochev, Gunzburger, *SIAM J. NUM. ANAL.* 2005

For $\phi_h \in P_k$ and $\mathbf{u}_h \in BDM_k$:

$$\|\phi - \phi_h\|_0 + \|\mathbf{u} - \mathbf{u}_h\|_0 = O(h^{k+1})$$

$$\|\phi - \phi_h\|_1 + \|\mathbf{u} - \mathbf{u}_h\|_{\text{div}} = O(h^k)$$

For $\phi_h \in P_k$ and $\mathbf{u}_h \in RT_k$:

$$\|\phi - \phi_h\|_0 + \|\mathbf{u} - \mathbf{u}_h\|_0 = O(h^k)$$

$$\|\phi - \phi_h\|_1 + \|\mathbf{u} - \mathbf{u}_h\|_{\text{div}} = O(h^k)$$

Velocity and pressure spaces need not form a stable mixed pair!



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Mimetic LS = Ritz Galerkin + Mixed Galerkin

Besides optimal accuracy, the Mimetic LS acquires valuable new properties:

Theorem Bochev, Gunzburger, *Least-squares finite element methods*, Springer, 2009

Assume that (ϕ^h, \mathbf{u}^h) solves the minimization problem

$$\min_{\phi^h \in G^h; \mathbf{u}^h \in D^h} \tilde{K}(\phi^h, \mathbf{u}^h) = \frac{1}{2} \left(\left\| \mathbf{A}^{-1/2} (\mathbf{u}^h + \mathbf{A} \nabla \phi^h) \right\|_0^2 + \left\| \gamma^{-1/2} (\nabla \cdot \mathbf{u}^h + \gamma \phi^h - f) \right\|_0^2 \right)$$

if $\gamma > 0$, (ϕ^h, \mathbf{u}^h) is **conservative** in the sense that there exists $\mathbf{w}^h \in C^h$; $\psi^h \in Q^h$ such that

$\Leftrightarrow (\phi^h, \mathbf{w}^h) \in G^h \times C^h$ solves the Ritz-Galerkin method and $\nabla \phi^h + \mathbf{w}^h = 0$

$\Leftrightarrow (\psi^h, \mathbf{u}^h) \in Q^h \times D^h$ solves the Mixed Galerkin method and $\nabla \cdot \mathbf{u}^h + \gamma \psi^h = \Pi^h f$

In other words, if 0th order term is present, the **mimetic least-squares** method computes

The same **pressure** approximation as in the **Ritz-Galerkin** method

The same **flux** approximation as in the **Mixed Galerkin** method



Computational proof ($\gamma=1$)

Example

$$p = -\exp x \sin y$$

$$\nabla \cdot \mathbf{u} = 0 \quad \text{in } \Omega$$

$$\mathbf{u} = -\nabla p$$

error	grid	16	32	64	128
$L^2 \mathbf{u}$	Mimetic LS	0.1514803E+00	0.7192623E-01	0.3523105E-01	0.1745720E-01
	Mixed	0.1514803E+00	0.7192623E-01	0.3523105E-01	0.1745720E-01
$H(\text{div})$	Mimetic LS	0.2869324E+01	0.1397179E+01	0.6894290E+00	0.3426716E+00
	Mixed	0.2869324E+01	0.1397179E+01	0.6894290E+00	0.3426716E+00

Flux: L^2 and $H(\text{div})$ errors of Mimetic LS and Mixed Galerkin identical

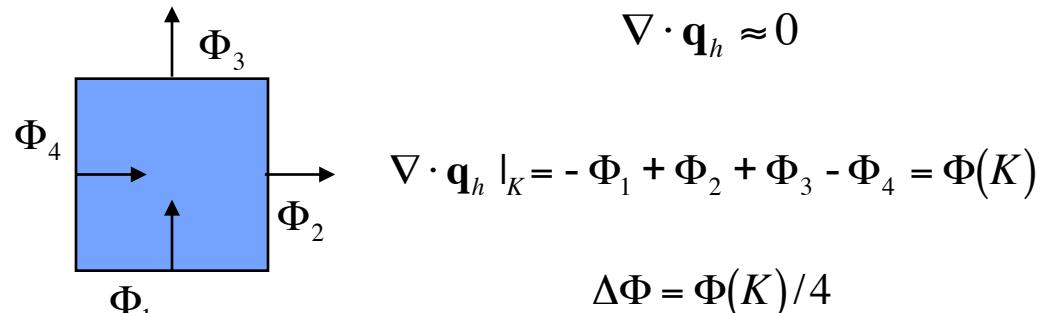
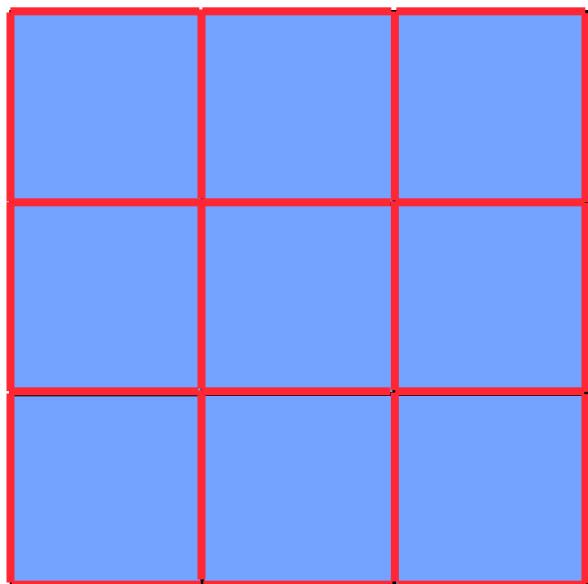


What happens with conservation when $\gamma=0$?

When $\gamma>0$, LS is **exactly conservative** $\rightarrow \gamma\psi_h = \Pi_h f - \nabla \cdot \mathbf{q}_h$

When $\gamma=0$, LS is **almost conservative** $\rightarrow 0 \approx \Pi_h f - \nabla \cdot \mathbf{q}_h$

A simple flux-correction procedure restores exact conservation

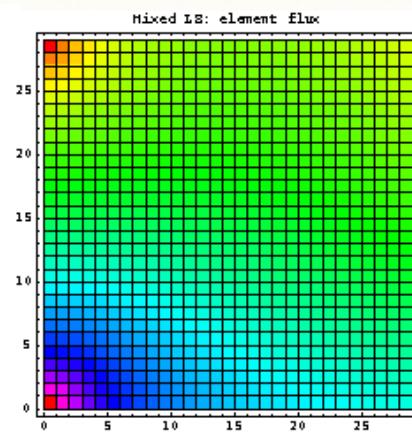
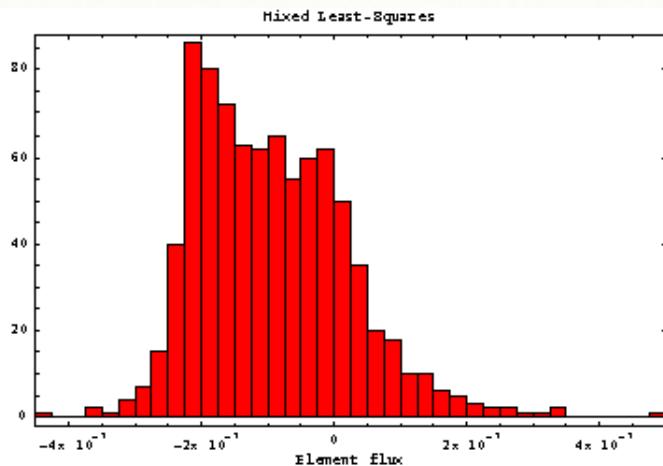


$$\nabla \cdot \tilde{\mathbf{q}}_h = 0$$

$$\begin{aligned}\tilde{\Phi}_1 &= \Phi_1 + \Delta\Phi & \tilde{\Phi}_3 &= \Phi_3 - \Delta\Phi \\ \tilde{\Phi}_2 &= \Phi_2 - \Delta\Phi & \tilde{\Phi}_4 &= \Phi_4 + \Delta\Phi\end{aligned}$$

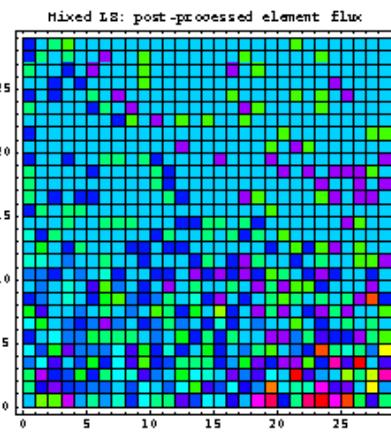
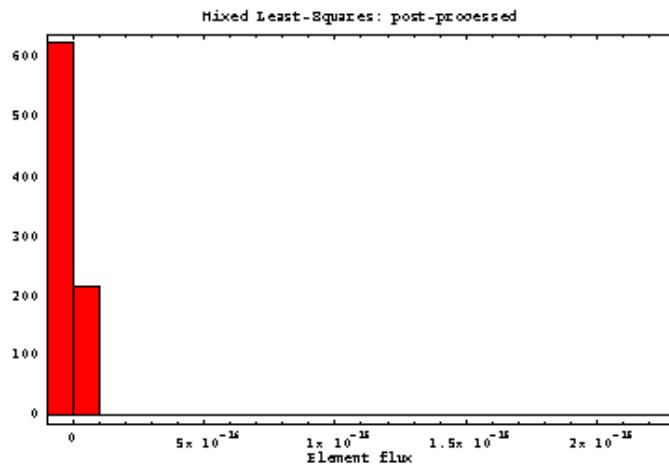


Flux Correction Does not Ruin Accuracy



Uniform grid

30x30 UNIF		L2	H(div)
MLS	0.2145816E-01	0.1235067E-03	
+FC	0.2145813E-01	0.4612541E-14	



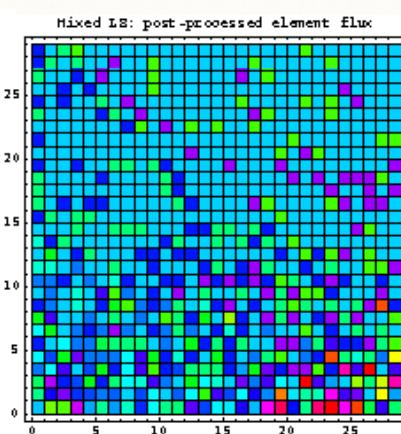
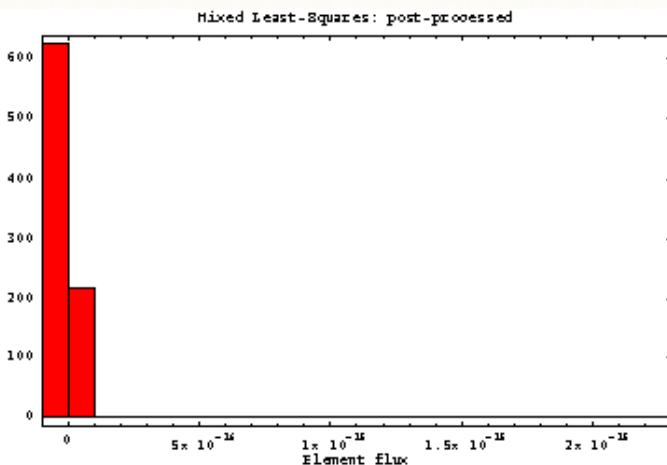
30x30 RAND		L2	H(div)
MLS	0.2342838E-01	0.1668777E-03	
+FC	0.2342836E-01	0.5281635E-14	

Random grid



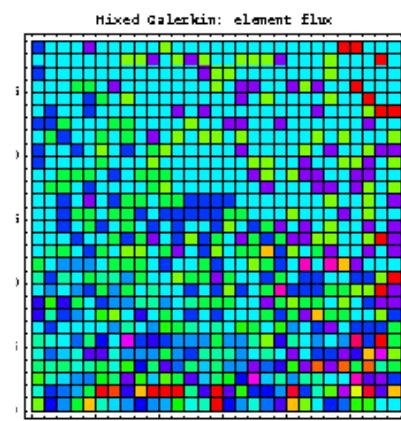
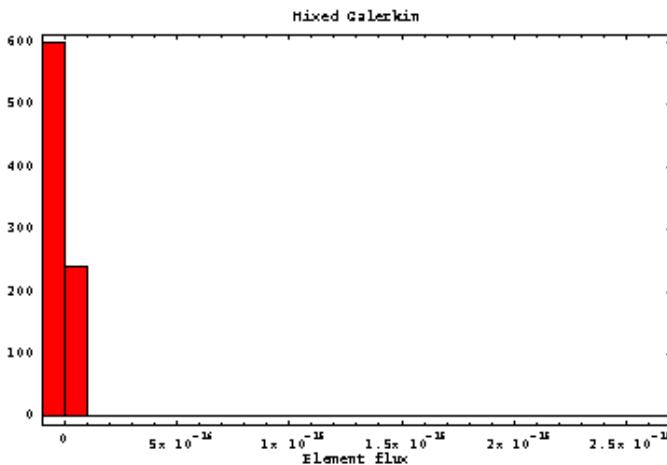
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Flux Corrected Mimetic LS vs. Mixed Galerkin



Uniform grid

30x30 UNIF		L2	H(div)
Mixed	0.2145805E-01	0.1446273E-13	
MLS+FC	0.2145813E-01	0.4612541E-14	



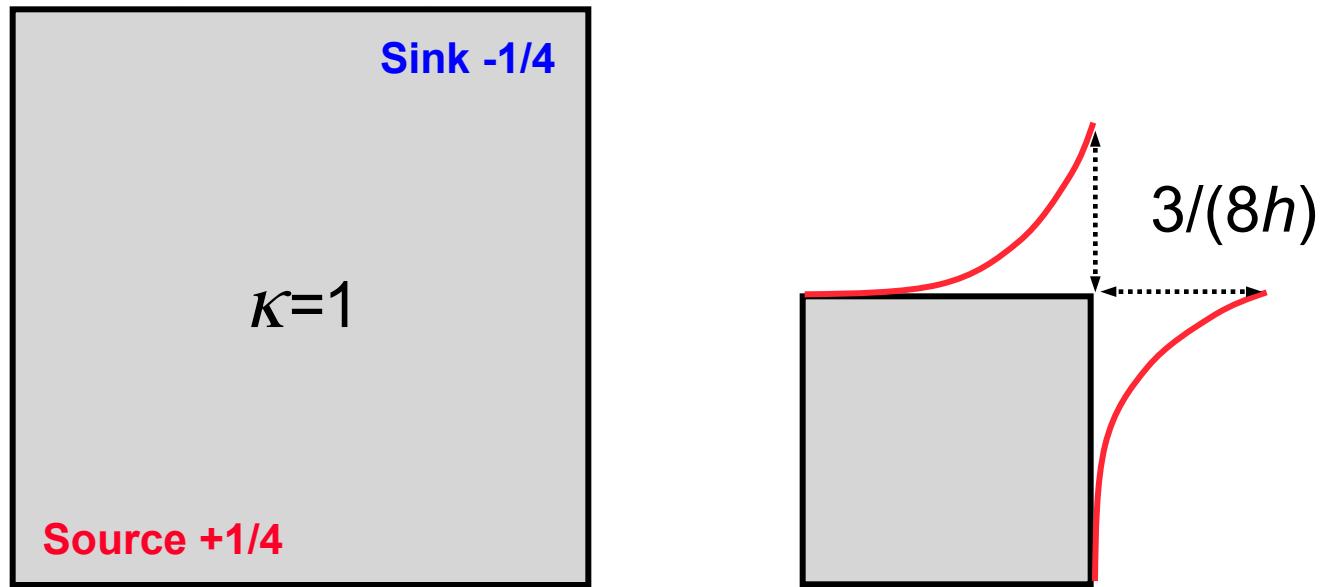
Random grid

30x30 RAND		L2	H(div)
Mixed	0.2352581E-01	0.6658244E-14	
MLS+FC	0.2342836E-01	0.5281635E-14	



The 5 Spot Problem

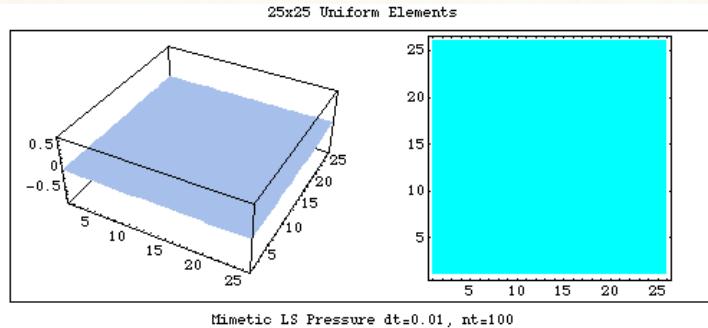
From: T. Hughes, A. Masud and J. Wan, A stabilized mixed DG method for Darcy flow



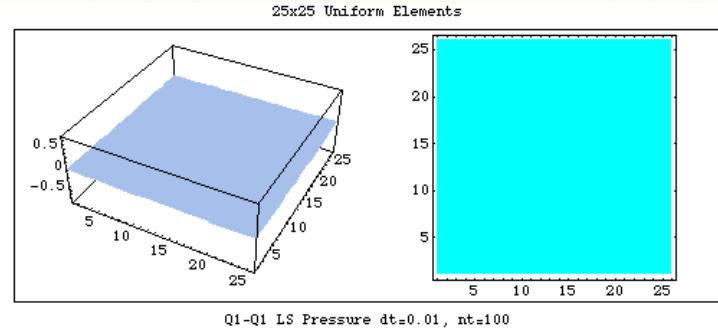
- ⇒ Problem is driven by a Neumann boundary condition ([normal flux](#))
- ⇒ Source/Sink is approximated by an equivalent distribution of the [normal flux](#)
- ⇒ Implicit Euler discretization in time
- ⇒ Grid has 625 uniform quad elements



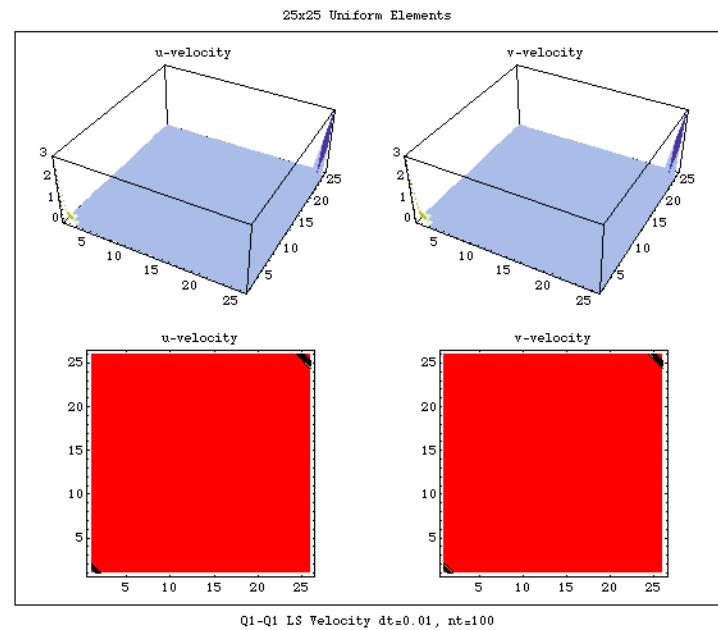
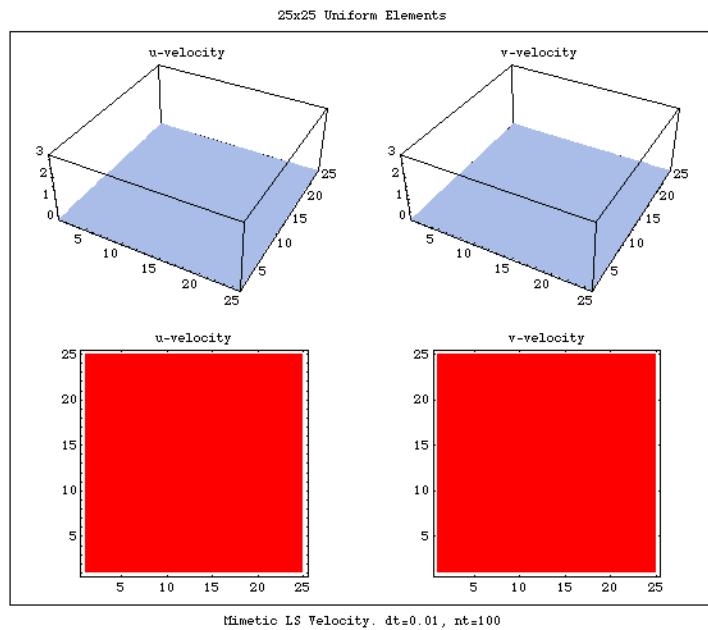
No Source Term



mimetic

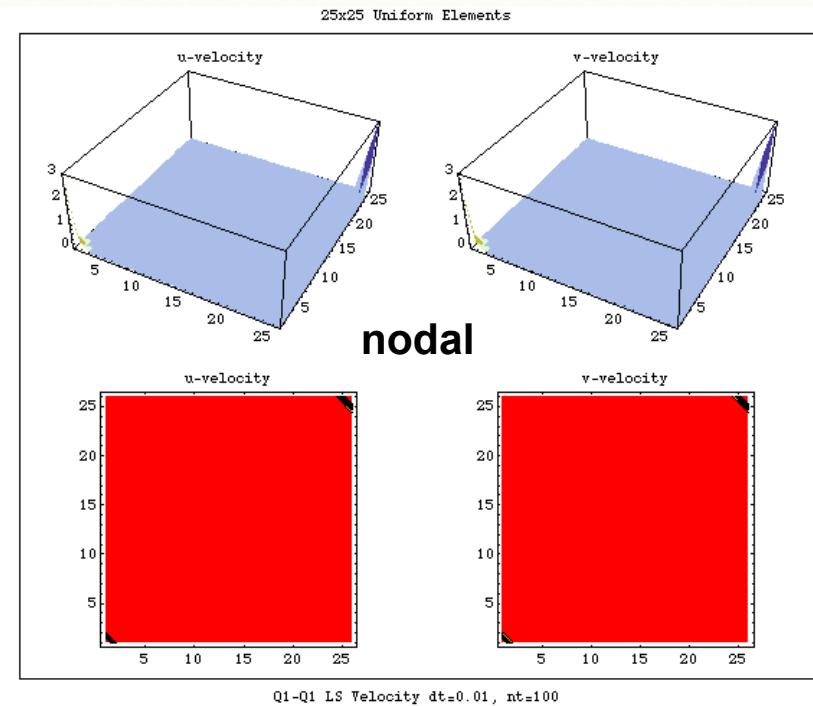
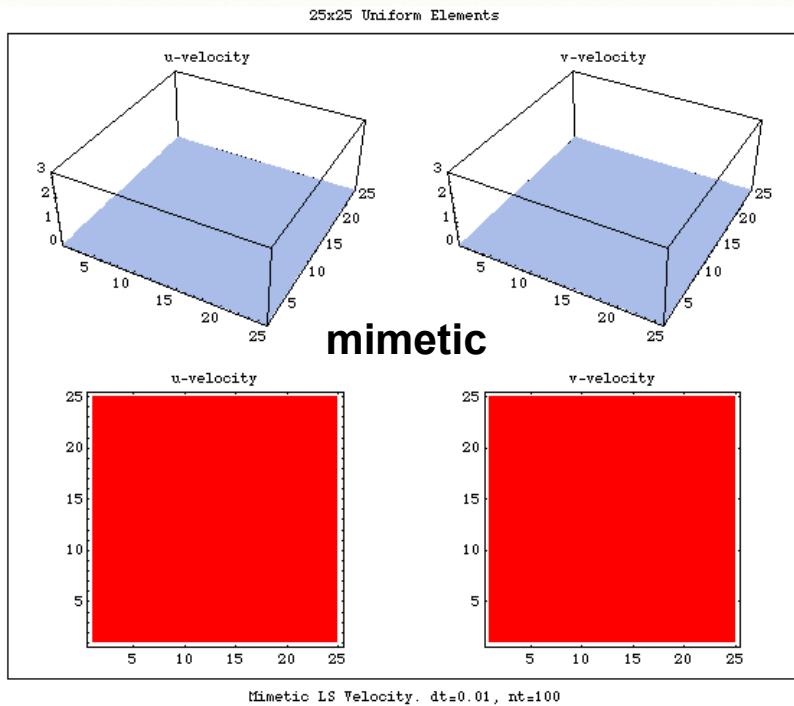


nodal



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Oscillatory Source



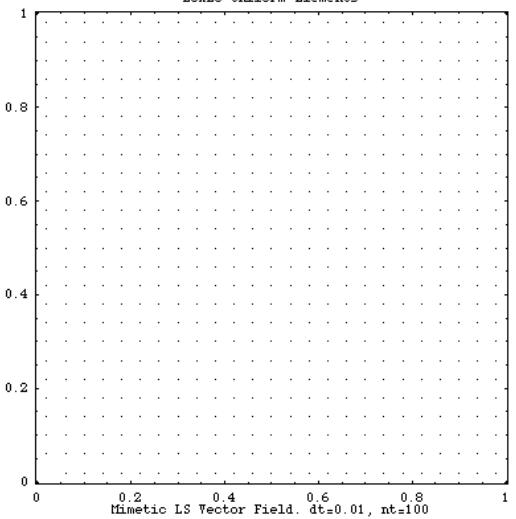
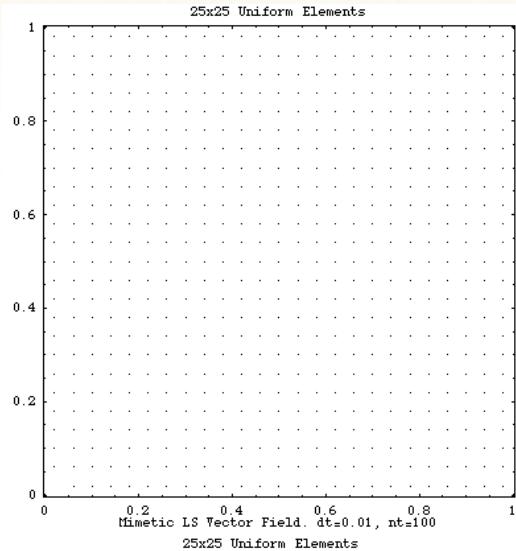
$$f = n \cos(\pi(n-1)x) \cos(\pi(n-1)y) \approx \frac{1}{\pi} \sqrt{\frac{\lambda}{2}} \varphi_{n,n}; \quad n = 25$$

$$\text{added perturbation} \approx \frac{1}{2\pi^2 n} |\varphi_{n,n}| \leq 0.002$$



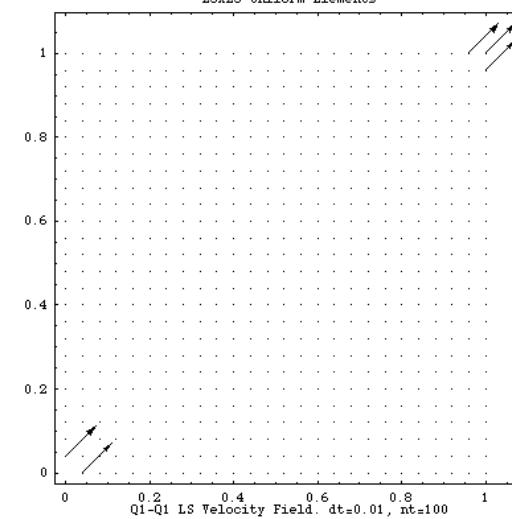
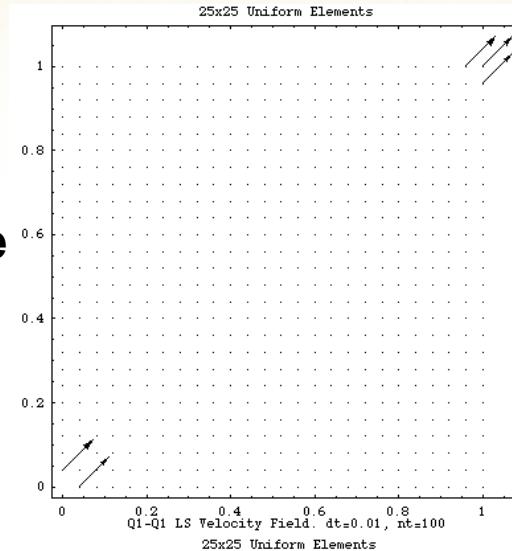
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Vector Field Comparison

Mimetic LS

**Source
OFF**

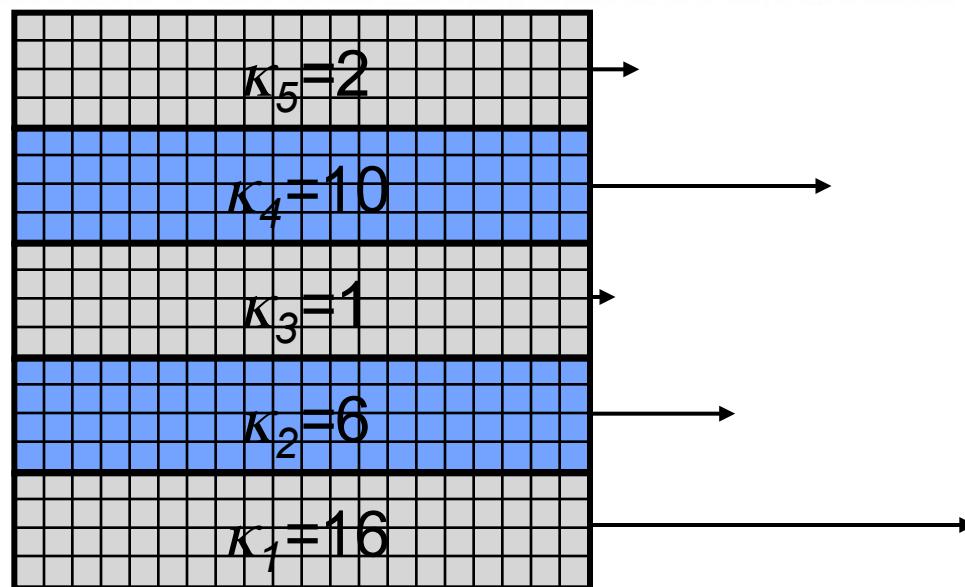
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Nodal LS

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The 5 Strip Problem

From: T. Hughes, A. Masud and J. Wan, *A stabilized mixed DG method for Darcy flow*



Exact solution

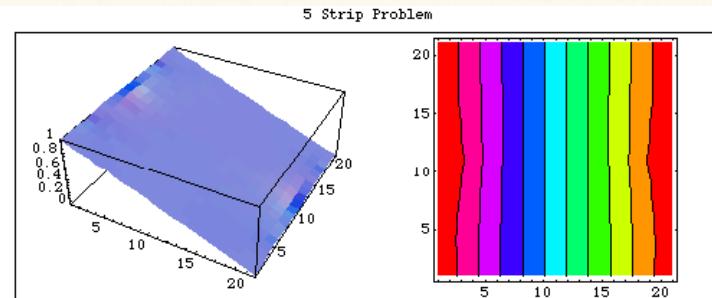
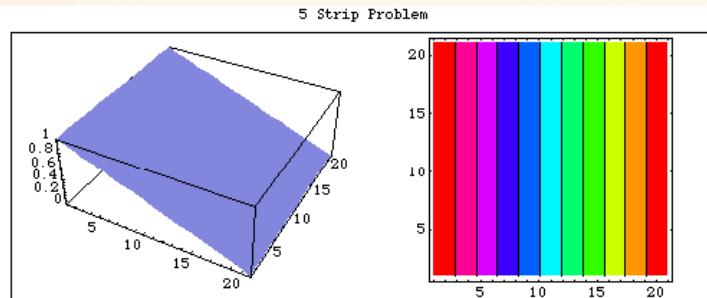
$$\phi = 1 - x;$$

$$\mathbf{u} = \begin{pmatrix} \kappa_i \\ 0 \end{pmatrix} \text{ in strip } i$$

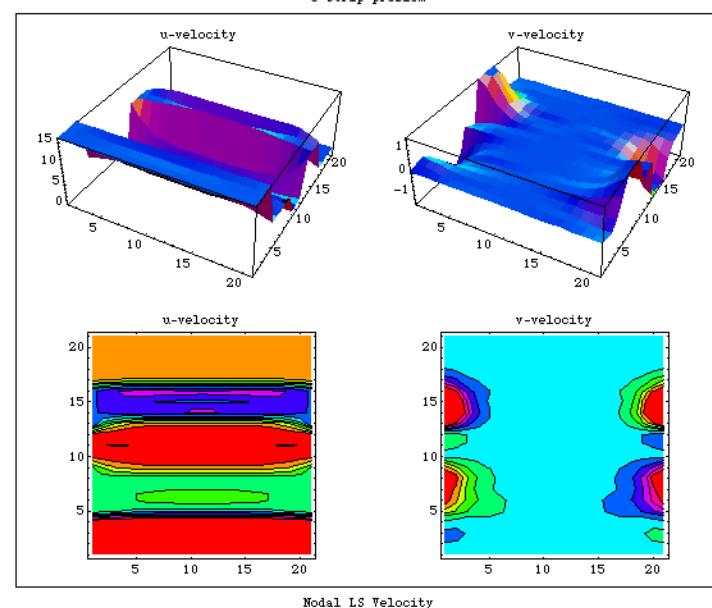
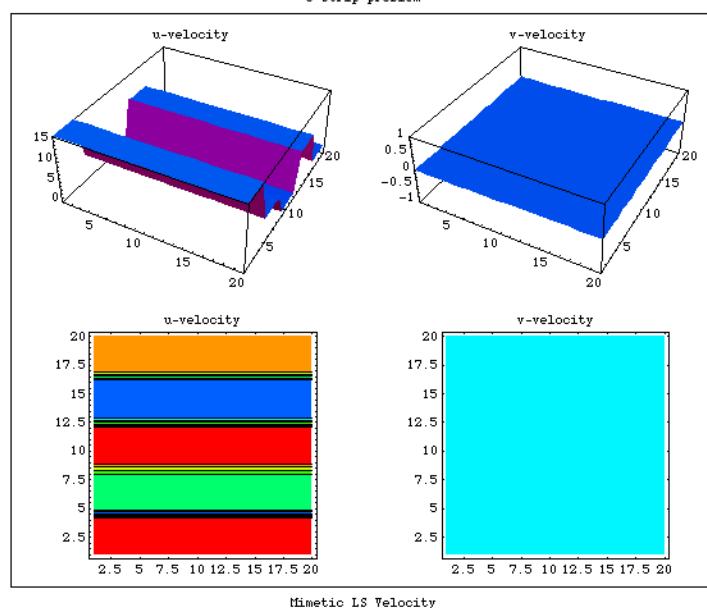
- ⇒ Problem is driven by Neumann boundary condition ([normal flux](#))
- ⇒ Implicit Euler discretization in time
- ⇒ Grid has 400 uniform elements [aligned with the interfaces](#) between the strips



Mimetic vs. Nodal Least Squares

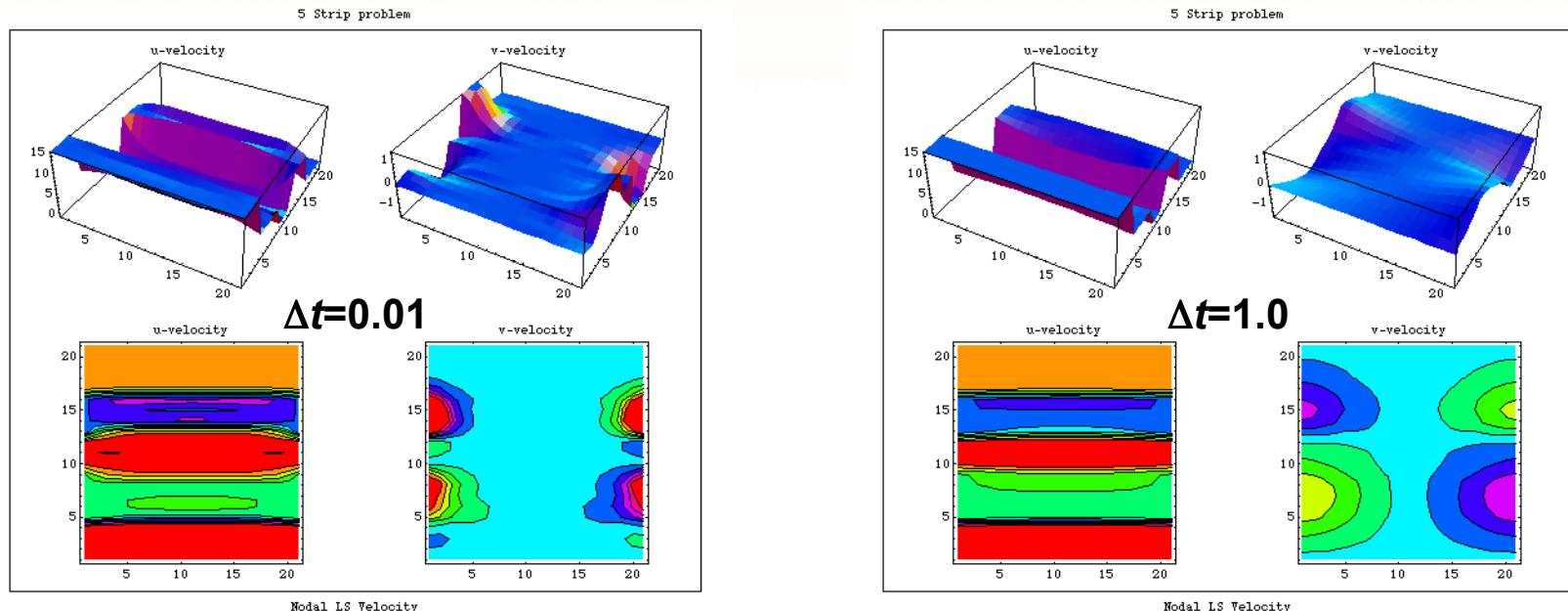


$\Delta t=0.01$



Method	L2 Flux	H(div) Flux	L2 Scalar	H1 Scalar
Mimetic LS	0.1670E-08	0.9839E-13	0.4553E-11	0.3041E-13
Nodal LS	0.1759E+01	0.7470E+02	0.8926E-02	0.1425E+00

Nodal LS at Different Time Steps



Nodal LS Solution worsens when Δt is reduced

Method	L2 Flux	H(div) Flux	L2 Scalar	H1 Scalar
$\Delta t = 1.0$	0.1925E+01	0.7206E+02	0.8892E-02	0.1423E+00
$\Delta t = 0.01$	0.1759E+01	0.7470E+02	0.8926E-02	0.1425E+00



Conclusions

Mimetic Least-Squares (MLS) offers important advantages:

- ✓ **discrete spaces** not subject to a **joint inf-sup**: can be selected **independently**!
- ✓ **MLS** inherit the **best** computational properties of Galerkin and Mixed methods:
 - Galerkin** → Optimal accuracy in the **pressure** variable
 - Mixed** → Optimal accuracy in the **flux** variable
- ✓ **MLS** are **locally conservative**
- ✓ **MLS** lead to **symmetric and positive definite** algebraic systems

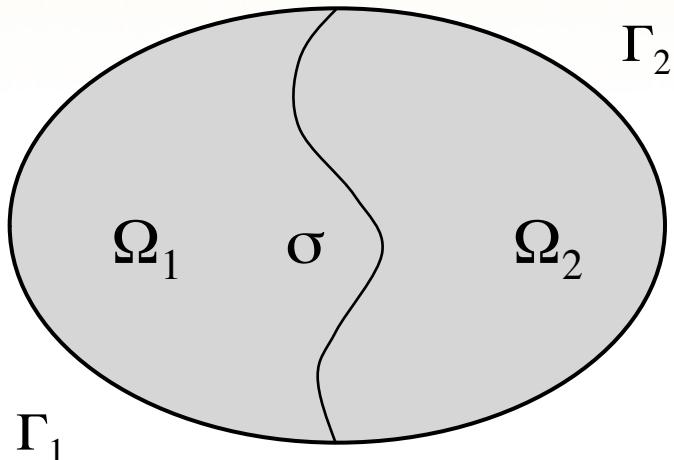
Mimetic least-squares are an (attractive) alternative to
mixed and finite volume schemes

But there's more!



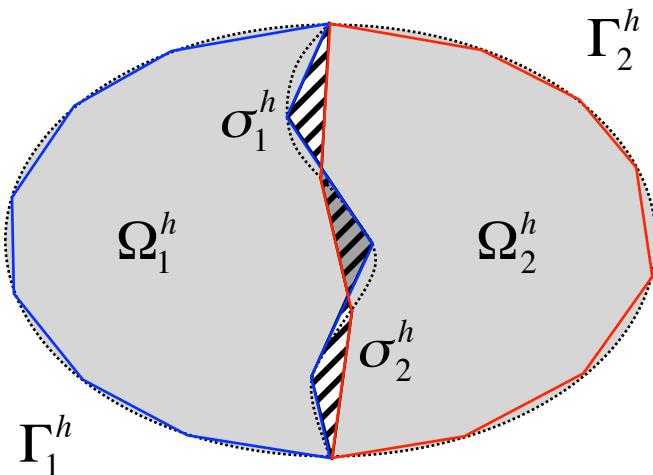
Bonus: Least-Squares for Mesh-Tying

Transmission problem



$$\begin{cases} \nabla \cdot \mathbf{u}_i + \gamma_i \phi_i = f & \text{in } \Omega_i \\ \mathbf{A}_i^{-1} \mathbf{u}_i + \nabla \phi_i = 0 & \text{in } \Omega_i \\ \phi_i = 0 & \text{on } \Gamma_{D,i} \\ \mathbf{n}_i \cdot \mathbf{u}_i = 0 & \text{on } \Gamma_{N,i} \end{cases} + \begin{cases} \phi_1 = \phi_2 \\ \mathbf{n}_1 \cdot \mathbf{u}_1 = \mathbf{n}_2 \cdot \mathbf{u}_2 \end{cases} \quad \text{on } \sigma$$

Discrete version



Approximation of **curved interfaces** leads to **non-matching** discrete interfaces and many problems:

- ⇒ traditional mortars not appropriate: duplicate interface
- ⇒ typically project values to one of the interfaces (master-slave)
- ⇒ issues with counting physical energy in gaps/voids
- ⇒ at best passes linear patch test (recovers linear pressure)
- ⇒ Non-matching interfaces remain a tough challenge for most traditional methods

Least-Squares Offer a Surprisingly Simple and Effective Solution

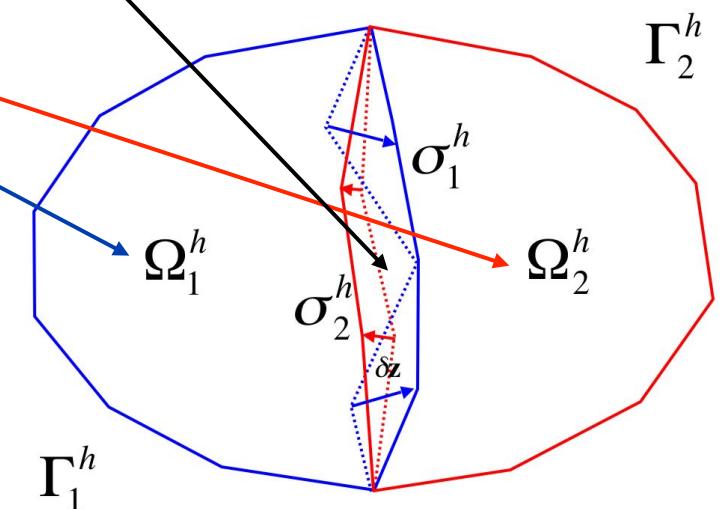
The reason:

- ⇒ LS are based on minimization of **artificial residual energy**, not physical energy
- ⇒ Minimization of **residual energy** allows to measure energy **redundantly**
- ⇒ All that is needed is elimination of the void regions to create sufficient overlap:
— Can be done by interface perturbation or by simply extending the domains

$$\min \frac{1}{2} \sum_i \underbrace{\left(\|\nabla \phi_i + \mathbf{v}_i\|_{0,\Omega_i}^2 + \|\nabla \cdot \mathbf{v}_i - f\|_{0,\Omega_i}^2 \right)}_{\text{residual energy}} + \underbrace{\|\phi_1 - \phi_2\|_{1,\Sigma}^2 + \|\mathbf{v}_1 - \mathbf{v}_2\|_{div,\Sigma}^2}_{\text{coupling term}}$$

Advantages

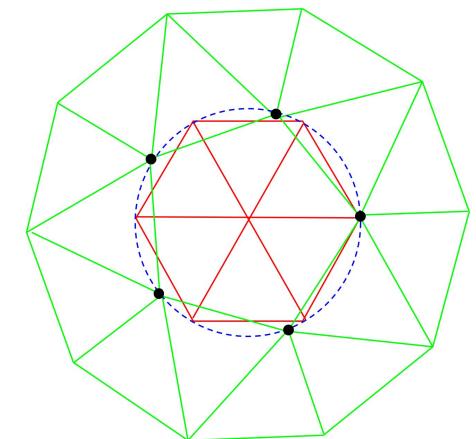
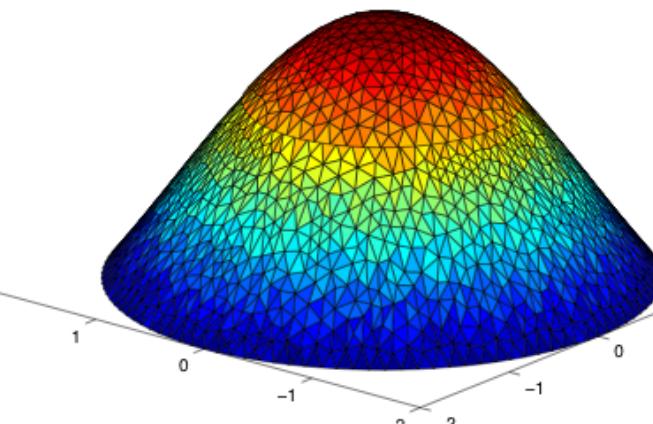
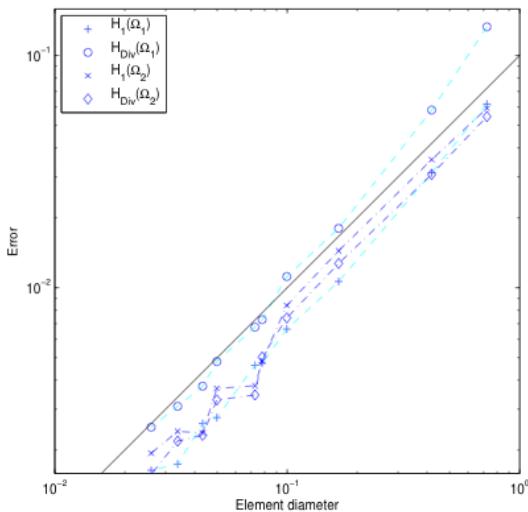
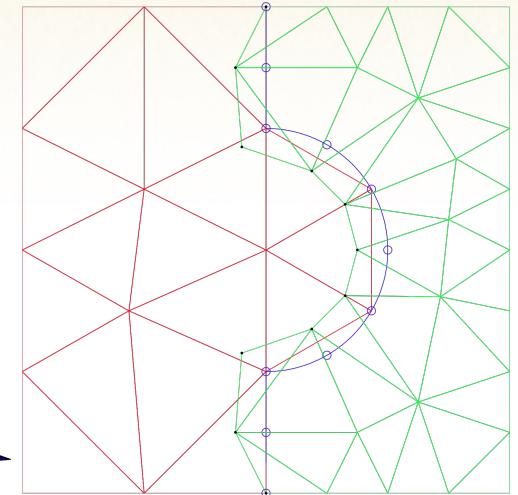
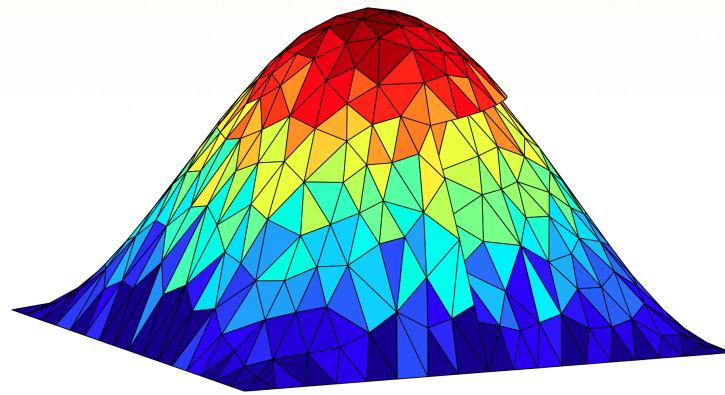
- ✓ Provably **stable** (coercive formulation)
- ✓ Provable **optimal convergence rate**
- ✓ Can pass an **arbitrary order** patch test
- ✓ No **mesh-dependant** tunable parameters
- ✓ No complications for **floating** domains
- ✓ Does not require extensive **mesh intersections**



Proof Of Concept: Doughnut & Circular Cut

Unstructured triangles

- ✓ theoretical rates achieved on both subdomains
- ✓ no ringing at interface nodes
- ✓ no complication from the floating domain!



Bochev, Day, JCAM 2007



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